

## SOLUTIONS OF EXERCISE SHEET 15

**Exercise 1.** Let  $f, g$  be meromorphic. Fix  $z_* \in U$ . Locally near  $z_*$  we can write

$$f(z) = \sum_{k=-n}^{\infty} a_k(z - z_*)^k, \quad g(z) = \sum_{k=-m}^{\infty} b_k(z - z_*)^k, \quad n, m \geq 0.$$

Thus,

$$f(z) + g(z) = \sum_{k=-\max\{n,m\}}^{\infty} (a_k + b_k)(z - z_*)^k,$$

$$f(z)g(z) = \sum_{k=-(n+m)}^{\infty} c_k(z - z_*)^k, \quad c_k = \sum_{\ell=-n}^{k+m} a_\ell b_{k-\ell},$$

are meromorphic functions is direct. To justify that  $1/f$  is also meromorphic it is enough to recall that the zeroes of  $f(z)$  are isolated (if the zeroes of  $f(z)$  were not isolated, then they would accumulate at some  $z_*$ , which would then imply that  $f \equiv 0$ ). Note also that if  $f(z)$  has a zero at  $z_* \in U$ , then, locally near  $z = z_*$ , we have that

$$f(z) = (z - z_*)^m h(z), \quad h(z_*) \neq 0,$$

for some  $m \in \mathbb{N}$  and some  $h$  holomorphic. Therefore

$$\frac{1}{f(z)} = (z - z_*)^{-m} \frac{1}{h(z)},$$

is meromorphic at  $z = z_*$ . The case in which  $f$  has a pole follows identically.

**Exercise 2.** First of all note that, since  $|g_n(z)| \leq \frac{1}{2}$  for all  $z \in U$ , then

$$\operatorname{Re}(1 + g_n(z)) \geq \frac{1}{2} > 0,$$

so  $1 + g_n(z)$  never hits the branch cut, and hence  $\log(1 + g_n(z))$  is well defined and holomorphic on  $U$ .

Regarding the normal convergence, there are multiple ways to justify it. For instance, one could write (for  $|w| \leq \frac{1}{2}$ )

$$\log(1 + w) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} w^n, \tag{0.1}$$

which in turn implies that

$$|\log(1 + w)| \leq |w| \sum_{n=0}^{\infty} \frac{1}{n+1} |w|^n \leq |w| \sum_{n=0}^{\infty} |w|^n \leq 2|w|, \tag{0.2}$$

having used the convergence of the geometric series for  $|w| \leq \frac{1}{2}$ . Then, using this last bound, substituting  $w$  by  $g_n(z)$ , and recalling that by hypothesis the sum of the  $g_n(z)$  converge normally, we conclude that

$$\sum_{n=1}^{\infty} \sup_{z \in K} |\log(1 + g_n(z))| \leq 2 \sum_{n=1}^{\infty} \sup_{z \in K} |g_n(z)| < \infty. \quad (0.3)$$

This concludes the proof. In case one doesn't recall the power series expansion in (0.1), the bound in (0.2) can also be obtained by noting that

$$\log(1 + w) = \int_0^1 \frac{w}{1 + tw} dw$$

and hence,

$$|\log(1 + w)| \leq |w| \int_0^1 \frac{1}{|1 + tw|} dw \leq 2|w|,$$

and then proceed as before.

**Exercise 3.** To prove local uniform convergence of the sequence of functions  $\prod_{k=1}^n (1 + g_k(z))$ , we only need to show that on every compact subset  $K \subseteq U$  we have that the sequence convergence uniformly. Fix a compact set  $K \subseteq U$ . We can assume w.l.o.g. that  $|g_k(z)| < \frac{2}{3}$  for all  $k \in \mathbb{N}$  and  $z \in K$ , because we know that  $\sup_K |g_k| \xrightarrow{k \rightarrow \infty} 0$  and therefore if this is not the case only need to consider the sequence of  $k \geq m$  and  $\prod_{k=m}^n (1 + g_k(z))$  for which this is the case. If this sequence converges uniformly on  $K$  then also the original sequence (why?). Define

$$L_n(z) = \sum_{k=1}^n \log(1 + g_k(z)), \quad \text{and} \quad L(z) := \sum_{n=1}^{\infty} \log(1 + g_n(z)).$$

Note that

$$h_n(z) = \exp(L_n(z)) = \prod_{k=1}^n (1 + g_k(z)). \quad (0.4)$$

Now, from the previous exercise we infer that  $L_n$  converges normally to the holomorphic function  $L(z)$ . Indeed, if the  $g_n$  converge normally, there exists at  $N = N(K) \gg 1$  sufficiently large such that

$$\sup_{z \in K} |g_n(z)| \leq \frac{1}{2} \quad \text{for all} \quad n \geq N.$$

Then we can apply the previous part. Now, given that  $\exp(\cdot)$  is entire and  $L_n$  converges uniformly on compact sets, we conclude that

$$\exp(L_n(z)) \rightarrow h(z)$$

locally uniformly (estimate (0.3) gives a uniform bound on every compact, and  $\exp(\cdot)$  is continuous, hence uniformly continuous on compact sets, which suffices). Also

observe that we have  $h(z) = \exp(L(z))$ . If one prefers a more explicit proof, define

$$M_K := \exp\left(\sum_{n=1}^{\infty} \sup_{z \in K} |\log(1 + g_n(z))|\right).$$

For every  $\varepsilon > 0$  small, there exists  $N = N(K) \gg 1$  sufficiently large such that for all  $m \geq N$ ,

$$\sum_{k=m+1}^{\infty} \sup_{z \in K} |g_k(z)| \leq \min\left\{1, \frac{\varepsilon}{2e^2 M_K}\right\}.$$

We are going to prove that, for all  $n \geq m \geq N$ ,

$$\sup_{z \in K} |h_n(z) - h_m(z)| \leq \varepsilon.$$

Indeed, we have that

$$\begin{aligned} \sup_{z \in K} |h_n(z) - h_m(z)| &\leq \sup_{z \in K} |h_m(z)| \left| \prod_{k=m+1}^n (1 + g_k(z)) - 1 \right| \\ &\leq M_K \sup_{z \in K} \left| \prod_{k=m+1}^n (1 + g_k(z)) - 1 \right| \\ &\leq M_K \left( \exp\left(2 \sum_{k=m+1}^n \sup_{z \in K} |g_k(z)|\right) - 1 \right) \leq \varepsilon, \end{aligned}$$

having used that  $e^x - 1 \leq xe^x$ .

The last part of the exercise can be proven by a double inclusion. If  $(1 + g_k(z_0)) = 0$  for some  $k$ , then  $h_n(z_0) = 0$  for all  $n \geq k$ , and hence  $h(z_0) = 0$ . For the other direction, let  $z_0 \in U$  be such that  $h(z_0) = 0$ . Take a small closed disc  $K = \overline{D(z_0, r)} \subset U$ . Let  $N = N(K) \gg 1$  sufficiently large so that  $|g_k(z)| < \frac{2}{3}$  on  $K$  for all  $k \geq N$ . By the convergence proof we know that

$$h(z) = \left( \prod_{k=1}^{N-1} (1 + g_k(z)) \right) \cdot \exp\left(\sum_{k=N}^{\infty} \log(1 + g_k(z))\right) =: F(z) \cdot T(z).$$

We know that the tail  $T(z) \neq 0$  for all  $z \in K$ . Therefore,  $F(z_0) = 0$ , and hence  $1 + g_k(z_0) = 0$  for some  $k \leq N - 1$ . The proof is complete.